MPRi – Computational Geometry and Topology

Homology and Persistent Homology

X : topo. space. $\in \mathrm{Top}$ $\quad k$: field $\quad x \xrightarrow{f} y$

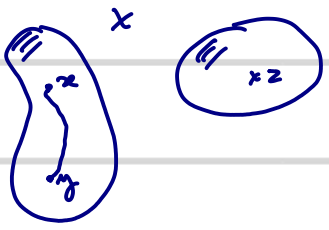
$\downarrow H_*(-;k)$ (homology) $\quad \downarrow H_*$

$H_*(X;k) \in \mathrm{Vec}_k \quad H_*(x;k) \xrightarrow{f_*} H_*(y;k)$

$H_* : \mathrm{Top} \longrightarrow \mathrm{Vec}_k$ s.t.:

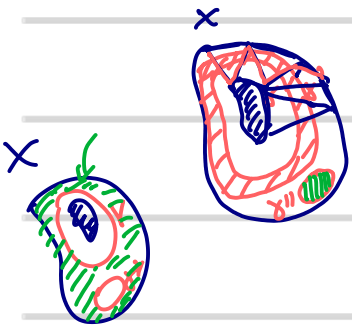
- $H_*(\mathrm{id}_X) = \mathrm{id}_{H_*(X)}$
- $(X \xrightarrow{f} Y \xrightarrow{g} Z) \xmapsto{H_*} (H_*(X) \xrightarrow{H_*(f)} H_*(Y) \xrightarrow{H_*(g)} H_*(Z))$ s.t. $H_*(g \circ f) = H_*(g) \circ H_*(f)$ ($g \circ f$; $H_*(g \circ f)$)

Intuition :



$x \sim y$ iff $\exists \gamma : [0,1] \longrightarrow X$ cont. s.t. $\gamma(\{0,1\}) = \{x, y\}$. ("$\partial\gamma$")

$\pi_0(X) := X/\sim$



$\gamma : S^1 \longrightarrow X$

$[0,1] \longrightarrow X$ s.t. $\gamma(\{0,1\}) = \{\cdot\}$.

$\gamma \sim \gamma'$ iff $\gamma \cup \gamma' = \partial(\Sigma \subseteq X)$

$= "\partial(\Gamma \Rightarrow X)"$

Simpl. Cplxes:

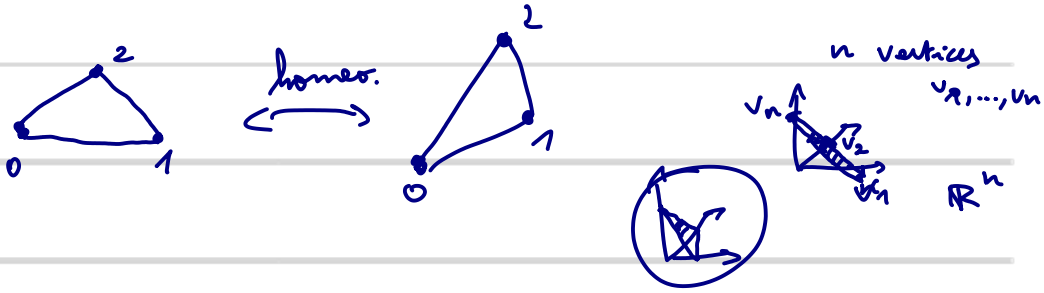
Def: V: set of vertices

$K \subseteq 2^V$: set simplices s.t.:

- $\forall \sigma \in K$, $\#\sigma$ is finite
- $\emptyset \notin K$
- $\forall \underset{\neq \emptyset}{\tau} \subseteq \sigma \subseteq V$, $\sigma \in K \Rightarrow \tau \in K$

Ex: $V = \{0, 1, 2\}$

$K = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$



Prop:
- for any simpl. cplx, there exist geometric realizations ("drawings" in topological spaces)
- all geometric realizations of K are homeomorphic.

↳ $|K|$: underlying (topo.) space of K.

Homology groups:

K: simplicial cplx.

k: field (char $k = 2$ $(1+1=0)$) — $k = \mathbb{Z}/2\mathbb{Z}$

$n \in \mathbb{Z}$: degree in homology.

Def: An n-chain in K is a formal k-linear combination of n-simplices of K:

$$c = \sum_{i=0}^{l} \alpha_i \sigma_i$$

$\alpha_i \in k$, $\sigma_i \in K_n = \{n\text{-simplices in } K\}$ $\{v_0, \dots, v_n\}$

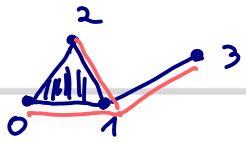
$c : K_n \to k$ such that supp. c if finite

$$\sum_{\sigma \in K_n} c(\sigma) . \sigma$$

$\sigma \in K_n \mapsto \begin{cases} \alpha_i \text{ if } \sigma = \sigma_i \text{ for some } i \\ 0 \text{ otherwise} \end{cases}$

$n = 1$:

$n = 2$:



$K =$

$n = 1$:

$k = \mathbb{Z}/2\mathbb{Z}$

$k = \mathbb{R}$

$c = \{0,1\} + \{1,3\} + \{2,1\}$

$e \mapsto \begin{cases} 1 \text{ if } e \in \{\{0,1\},\{1,3\},\{2,1\}\} \\ 0 \text{ o/w} \end{cases}$

$c = 0.7\{0,1\} + 3.9\{1,3\} - 7.2\{2,1\}$

$e \mapsto \begin{cases} 0.7 \text{ if } e = \{0,1\} \\ 3.9 \text{ if } e = \{1,3\} \\ -7.2 \text{ if } e = \{2,1\} = \{1,2\} \\ 0 \text{ o/w} \end{cases}$

Prop: $\{n\text{-chains on } K\}$ is a k-vector space:

$\forall\ c, c' : K_n \to k$ w/ finite supp., $\forall\ \alpha \in k$,

$c + c' : K_n \to k$ w/ finite supp.

$\alpha c : K_n \to k$ ——

$C_n(K; k) := \{n\text{-chains on } K\} = k^{(K_n)}$. That space is generated by $\{\mathbb{1}_\sigma,\ \sigma \in K_n\} = \{\sigma \mid \sigma \in K_n\}$

Def: n-boundary operator:

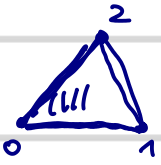
$$\partial_n : C_n(K;k) \longrightarrow C_{n-1}(K;k) \quad k\text{-linear}$$

$$\sigma \in K_n \longmapsto \sum_{v \in \sigma} \sigma \setminus \{v\} \in C_{n-1}(K;k)$$

$$\| \qquad\qquad \| \qquad\qquad \leftarrow$$

$$\{v_0, \dots, v_n\} \longmapsto \sum_{i=0}^{n} (-1)^i \{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$$

$$\{v_0, \dots, \hat{v}_i, \dots, v_n\}$$



$$\partial_1 \{0,2\} = \{2\} + \{0\}$$

$$\partial_2 (\{0,1,2\}) = \{1,2\} + \{0,2\} + \{0,1\}$$

$$c = \sum_{j=1}^{l} \alpha_j \cdot \sigma_j \in C_n(K;k), \quad \sigma_j \in K_n$$

$$\hookrightarrow \partial_n c := \sum_{j=1}^{l} \alpha_j \cdot \partial_n \sigma_j .$$

Def:

- An n-cycle is an n-chain c s.t. $\partial_n c = 0$.
  $\hookrightarrow \ker \partial_n =: \{n\text{-cycles in } K\}$.
- An n-boundary is an n-chain c s.t. $c = \partial_{n+1} c'$ for some $c' \in C_{n+1}(K;k)$
  $\hookrightarrow \operatorname{Im} \partial_{n+1} =: \{n\text{-boundaries in } K\}$

Def: n^{th} homology group of K over k:

$$H_n(K;k) := \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

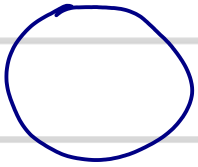
Caveat: require $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$.
i.e. $\partial_n \circ \partial_{n+1} = 0$.

Prop: When $\mathrm{char}\, k = 2$, we have:

$$\forall n \in \mathbb{N},\quad \partial_n \circ \partial_{n+1} = 0.$$

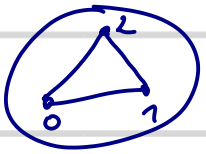
$$[\ \text{——}\ ] + [\ \text{——}\ ] = 0$$

$$2 \cdot [\ldots] = 0$$

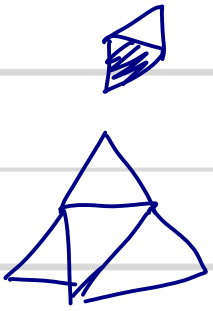


homeo.

$\simeq$



$K = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}\}$



$C_0(K) = \langle \{0\}, \{1\}, \{2\} \rangle \simeq k^3$

$C_1(K) = \langle \{0,1\}, \{1,2\}, \{0,2\} \rangle \simeq k^3$

$C_n(K) = 0 \quad \forall n \geq 2$

$\beta_1 = 5$

$\beta_0 = 2$

$\partial_0 = 0$ $\qquad \partial\{v\} = \sum_v \{v\} \setminus \{v\} = 0.$

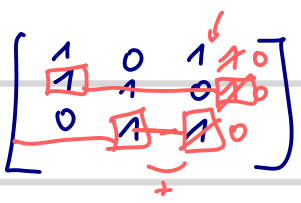
$$\partial_1 = \begin{array}{c} \\ \{0\} \\ \{1\} \\ \{2\} \end{array} \begin{array}{c} \begin{array}{ccc} \{0,1\} & \{1,2\} & \{0,2\} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$\partial_2 = 0$

$k = \mathbb{Z}/2\mathbb{Z}$

$\mathrm{rk}\, \partial_0 = \mathrm{rk}\, \partial_2 = 0$

$\mathrm{rk}\, \partial_1 = ?$



$\mathrm{rk}\, \partial_1 = 2 \qquad \dim \ker \partial_1 = 1$

$$H_n(K) := \ker \partial_n / \mathrm{Im}\, \partial_{n+1}$$

$$\dim H_n(K) = \dim(\ker \partial_n) - \dim(\mathrm{Im}\, \partial_{n+1}) \quad (= \beta_n \text{ } n\text{-th Betti number})$$

$$= \#K - \mathrm{rk}\, \partial_n - \mathrm{rk}\, \partial_{n+1}$$

$\dim H_0(K) = 3 - 0 - 2 = 1$

$\dim H_1(K) = 3 - 2 - 0 = 1.$

Can extend H_* to:

- fields of arbitrary characteristics
- arbitrary topo. spaces.
- continuous maps $X \xrightarrow{f} Y$ s.t. $\begin{cases} H_*(g \circ f) = H_*(g) \circ H_*(f) \\ H_*(id_X) = id_{H_*(X)} \end{cases}$

Prop: $X \overset{\text{homeo.}}{\simeq} Y \Rightarrow H_*(X; k) \overset{(\text{iso.})}{\simeq} H_*(Y; k)$

proof: Let $h : X \to Y$ be a homeo.

Consider $h^{-1} : Y \to X$

$$\begin{cases} id_{H_*(X)} = H(id_X) = H_*(h^{-1} \circ h) = H_*(h^{-1}) \circ H_*(h) \\ id_{H_*(Y)} = H_*(id_Y) = H_*(h \circ h^{-1}) = H_*(h) \circ H_*(h^{-1}) \end{cases}$$

∎

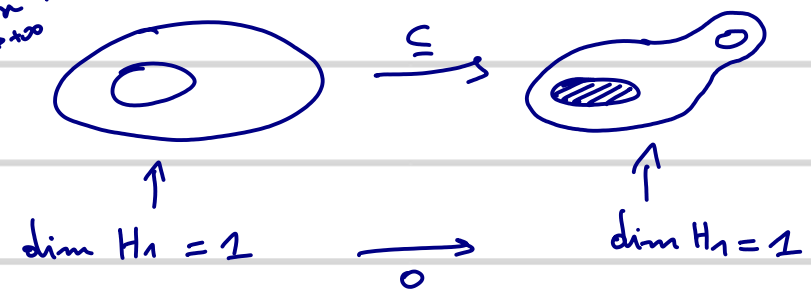
Persistent Homology:

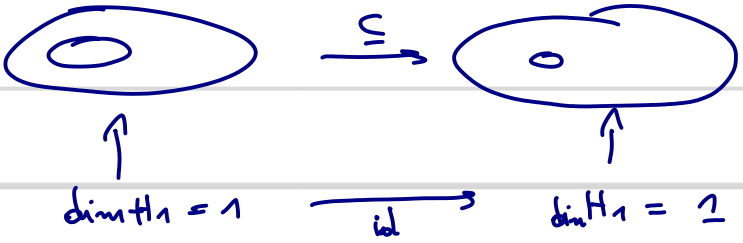
$f : X \to \mathbb{R}$

$\forall t \in \mathbb{R}, \quad F_t := f^{-1}((-\infty, t])$.

$\{F_t\}_{t \in \mathbb{R}}$ together with inclusion maps $\{F_t \subseteq F_{t'}\}_{t \leq t' \in \mathbb{R}}$.

↑ filtration of $X = \lim_{t \to +\infty} F_t$





Pf: A filtration is a family of topo. spaces $\{F_t\}_{t \in \mathbb{R}}$ that is nested w.r.t. inclusion: $F_t \subseteq F_{t'}$ $\forall t \leq t' \in \mathbb{R}$.

$\downarrow H_*(-; k)$

Def: persistence module

$\{H_*(F_t; k)\}_{t \in \mathbb{R}}$ of k-vector spaces together with $\{H_*(F_t; k) \xrightarrow{f_{t,t'}} H_*(F_{t'}; k)\}_{t \leq t' \in \mathbb{R}}$ of k-linear maps

s.t.:

- $\forall t \in \mathbb{R}, \quad f_{t,t} = \mathrm{id}_{H_*(F_t; k)}$
- $\forall t \leq t' \leq t'' \in \mathbb{R}, \quad f_{t,t''} = f_{t',t''} \circ f_{t,t'}$.

Can define:

- zero module: $\{0\}_{t\in\mathbb{R}}$, $\{0: 0\to 0\}_{t\le t'\in\mathbb{R}}$
- direct sums: $M=\{M_t\}_{t\in\mathbb{R}}$, $\{m_{t,t'}\}_{t\le t'\in\mathbb{R}}$; $N=\{N_t\}_{t\in\mathbb{R}}$, $\{n_{t,t'}\}_{t\le t'\in\mathbb{R}}$

$$(M\oplus N)_t := M_t \oplus N_t$$

$$(M\oplus N)_t \longrightarrow (M\oplus N)_{t'}$$

$$m_{t,t'}\oplus n_{t,t'} = \begin{bmatrix} m_{t,t'} & 0 \\ 0 & n_{t,t'} \end{bmatrix}$$

- morphisms $M\to N$: $\{\varphi_t\}_{t\in\mathbb{R}}$ s.t. $\forall\, t\le t'\in\mathbb{R}$:

$$\begin{array}{ccc} M_t & \xrightarrow{m_{t,t'}} & M_{t'} \\ \downarrow{\scriptstyle \varphi_t} & \circlearrowright & \downarrow{\scriptstyle \varphi_{t'}} \\ N_t & \xrightarrow[n_{t,t'}]{} & N_{t'} \end{array}$$

- $\ker\varphi$, $\operatorname{Im}\varphi$, $\operatorname{coker}\varphi$, etc.
- $\ker\varphi=0 \iff \varphi_t$ is inj. $\forall t$ (monomorphism)
- $\operatorname{coker}\varphi=0 \iff \varphi_t$ is surj. $\forall t$ (epimorphism)
- $\ker\varphi=\operatorname{coker}\varphi=0 \iff \varphi_t$ is an iso. $\forall t$ (isomorphism).

Def: M is decomposable if $M \overset{(\text{iso.})}{\simeq} N\oplus U$ where $N, U\neq 0$.

M is indecomposable otherwise.

Thm: Under proper (and mild) conditions, any persistence module M decomposes into a direct sum of indecomposable modules: $M\simeq \bigoplus_{j\in J} M_j$ s.t. each M_j is indecomposable.

Q What do these indecomposables look like?

Prop: The indecomposables are "interval modules":

I: interval of $\mathbb{R}$

$$\left(k_I\right)_t := \begin{cases} k & \text{if } t \in I \\ 0 & \text{elsewhere} \end{cases}$$

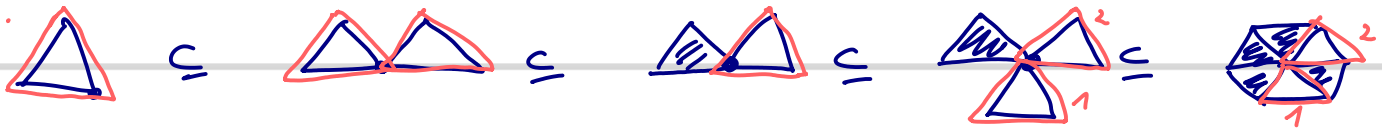
$$\left((k_I)_t \to (k_I)_{t'} \right) := \begin{cases} id_k & \text{if } t, t' \in I \\ 0 & \text{otherwise.} \end{cases}$$

Thm + Prop $\Rightarrow$ $\forall$ pers. mod. M (with proper conditions):

$$M \cong \bigoplus_{j \in J} k_{I_j}$$ where k_{I_j} is the interval module supported on $I_j \subseteq \mathbb{R}$.

And this decomposition is unique $\rightsquigarrow$ **Def:** $\mathcal{B}(M) := \{ I_j \mid j \in J \}$.

Filt. →



Pers. module →

$$k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} k \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}} k^2$$